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Another Leigh-Strassler deformation through the matrix model

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Abstract: In here the matrix model approach, by Dijkgraaf and Vafa, is used in order to obtain the effective superpotential for a certain deformation of $N = 4$ SYM discovered by Leigh and Strassler. An exact solution to the matrix model Lagrangian is found and is expressed in terms of elliptic functions.

Keywords: Supersymmetric Effective Theories, Matrix Models.

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Contents

1. Introduction

Recently Dijkgraaf and Vafa proposed a method using the matrix model to calculate the effective superpotential for $N = 1$ supersymmetric gauge theories [1]–[4]. In particular they showed for the mass perturbation of $N = 4$ SYM theory down to an $N = 1$ theory, that the effective superpotential agreed with the result by Dorey [5] obtained with different methods.

One class of interesting $N = 1$ superconformal theories are the ones discovered by Leigh and Strassler as marginal and relevant deformations of $N = 4$ SYM. We use the matrix model to extract the effective superpotential for a particular relevant perturbation of $N = 4$ SYM, discovered by Leigh and Strassler [6]. This superpotential is very similar to the one studied by Dorey et.al. [7], it differs though through the fields in the q-deformed commutator, in their case they are $\Phi^+ = \Phi_1 + i\Phi_2$ and $\Phi^- = \Phi_1 - i\Phi_2$ and in our case we have Φ_1 and Φ_2 therefore it could be of interest to really understand the physical differences between them. In the case when the deformed commutator becomes an ordinary commutator they both reduce to the mass deformed theory mentioned above.

Even though this case looks at first sight a bit more difficult than the case studied by Dorey etc, it has some nice properties that will be seen. For instance it transforms in a nice way under S transformation and the eigenvalue distribution is symmetrical distributed around zero. We expect that these nice properties are due to the symmetries between exchange of the fields which is absent in the other [7] case.

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2. The action and the elliptic world

The superpotential of the Leigh-Strassler deformed SYM that will be investigated looks like

$$W = -\operatorname{Tr}\left(e^{i\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\beta}\Phi_1\Phi_3\Phi_2 - m\sum\Phi_i^2\right) \tag{2.1}$$

where the fields Φ_i are chiral transforming under $\mathrm{U}(N)$. Now we would like to apply the method of Dijkgraaf and Vafa in order to find an effective superpotential of the $\mathrm{U}(N)$ theory in its confining vacua. Accordingly we should look at the partition function

$$Z = \int \mathcal{D}\Phi e^{\frac{1}{g_s}\operatorname{Tr}\left(e^{i\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\beta}\Phi_1\Phi_3\Phi_2 - m\sum\Phi_i^2\right)} \tag{2.2}$$

and expand it around the classical vacuum where $\Phi_{cl} = 0$. The fields are all to be treated as hermitian matrices. It would be nice to see that even in this case the equations of motions for the quantum fluctuation will yield an elliptic structure, and indeed we will see that that is the case. Integrating out Φ_2 and Φ_3 and diagonalizing Φ_1 the partition function looks like

$$\int \prod d\lambda_i \prod_{j<i} \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i + \lambda_j)^2 \sin^2\beta + (\lambda_i - \lambda_j)^2 \cos^2\beta + 4m^2} \prod_i \frac{e^{-\frac{m}{g_s}\lambda_i^2}}{\sqrt{\lambda_i^2 \sin^2\beta + m^2}}\,. \tag{2.3}$$

Here we see that the effective action look like something with an external quadratic potential well and the numerator as a coulomb force for electrons moving in two dimensions, the denominator looks a bit more tricky with both a repulsive part and an attractive. The classical minimum will be in the minimum for the potential well, which is $\lambda = 0$. There is no force pulling it more in the negative than the positive direction, so in the quantum regime the eigenvalues will be distributed symmetrically around $\lambda = 0$. Making the following change in coordinates

$$\lambda_i = 2m\frac{\sinh\mu_i}{\sin 2\beta} \tag{2.4}$$

the partition function transforms into

$$\begin{aligned}\int \prod d\mu_i \prod_{j<i} &\frac{(\sinh\mu_i - \sinh\mu_j)^2}{(\sinh\mu_i - \sinh(\mu_j + 2i\beta))(\sinh\mu_i - \sinh(\mu_j - 2i\beta))} \times \\ &\times \prod_i \frac{e^{-4m^3\sinh^2\mu_i/g_s\sin^2 2\beta}\cosh\mu_i}{\sqrt{\sinh\mu_i - \sinh(\mu_i + 2i\beta)}\sqrt{\sinh\mu_i - \sinh(\mu_i - 2i\beta)}}\end{aligned} \tag{2.5}$$

a bit of algebra makes it look like up to some constant:

$$\begin{aligned}\int \prod d\mu_i \prod_{j<i} &\frac{(\sinh(\frac{\mu_i-\mu_j}{2})\cosh(\frac{\mu_i+\mu_j}{2}))^2}{\sinh(\frac{\mu_i-\mu_j}{2} - i\beta)\cosh(\frac{\mu_i+\mu_j}{2} + i\beta)\sinh(\frac{\mu_i-\mu_j}{2} + i\beta)\cosh(\frac{\mu_i+\mu_j}{2} - i\beta)} \times \\ &\times \prod_i \frac{e^{-4m^3\sinh^2\mu_i/g_s\sin^2 2\beta}\cosh\mu_i}{\sqrt{\cosh(\mu_i + i\beta)}\sqrt{\cosh(\mu_i - i\beta)}}\,.\end{aligned} \tag{2.6}$$

Now we rescale the μ_i's with a factor of one half, because it will be convenient later on and calculate the equations of motions from the effective action

$$\begin{aligned}
&\frac{2m^3}{g_s \sin^2 2\beta}\sinh x - \frac{1}{2}\tanh\frac{x}{2} + \frac{1}{4}\tanh\left(\frac{x}{2} + i\beta\right) + \frac{1}{4}\tanh\left(\frac{x}{2} - i\beta\right) = \\
&\quad = \sum_j \left[\frac{1}{2}\coth\left(\frac{x-\mu_j}{4}\right) - \frac{1}{4}\coth\left(\frac{x-\mu_j}{4} + i\beta\right) - \frac{1}{4}\coth\left(\frac{x-\mu_j}{4} - i\beta\right)\right] + \\
&\qquad + \sum_j \left[\frac{1}{2}\tanh\left(\frac{x+\mu_j}{4}\right) - \frac{1}{4}\tanh\left(\frac{x+\mu_j}{4} + i\beta\right) - \frac{1}{4}\tanh\left(\frac{x+\mu_j}{4} - i\beta\right)\right]. \qquad (2.7)
\end{aligned}$$

Here x is one of the eigenvalues for which we will solve the equation of motion. We will use the fact that the eigenvalues λ_i are distributed symmetrical around $\lambda = 0$ then it implies that also μ_i will be distributed symmetrical around $\mu = 0$, thus they will take values between say $-a$ and a. A resolvent can then be introduced (see [8])

$$w(z) = \frac{1}{2}\int dx \frac{\rho(x)}{\tanh(\frac{z-x}{2})}\,, \qquad (2.8)$$

where $\rho(x) = \frac{1}{N}\sum \delta(x-\mu_i)$ and $w(-x) = -w(x)$. The resolvent will have a cut along the eigenvalues, $z \in [-a, a]$, and the jump in the resolvent along the cut will give the density

$$-2\pi i \rho(x) = w(x+i\epsilon) - w(x-i\epsilon) \qquad (2.9)$$

and the normalisation condition on the density can be expressed like

$$1 = \int_{-a}^{a} dx\rho(x) = \frac{1}{2\pi i}\oint_C w(x)\,, \qquad (2.10)$$

where C is a curve going around the cut. Then the equations of motion can be written like

$$\begin{aligned}
&\frac{2m^2}{\sin^2 2\beta}\sinh x - \frac{S}{2N}\tanh\frac{x}{2} + \frac{S}{4N}\tanh\left(\frac{x+2i\beta}{2}\right) + \frac{S}{4N}\tanh\left(\frac{x-2i\beta}{2}\right) = \\
&\quad = \frac{S}{N}\sum_j \left[\coth\left(\frac{x-\mu_j}{2}\right) - \frac{1}{2}\coth\left(\frac{x-\mu_j+4i\beta}{2}\right) - \frac{1}{2}\coth\left(\frac{x-\mu_j-4i\beta}{2}\right)\right] \qquad (2.11)
\end{aligned}$$

and the right hand side can be written in terms of the resolvent

$$S\left(2w(x) - w(x+4i\beta) - w(x-4i\beta)\right) \qquad (2.12)$$

here we introduced the 't Hooft coupling $S = g_s N$. This can in turn be written as

$$g(x+2i\beta) - g(x-2i\beta)\,, \qquad (2.13)$$

where

$$g(x) = w(x-2i\beta) - w(x+2i\beta) \qquad (2.14)$$

notice that $g(x)$ is periodic when x is shifted with $2\pi i$. We would also like to rewrite the left hand side in the same manner, with the same constant. Let us have a look at the following function

$$h(x) = -i\xi\cosh x + \frac{S}{4N}\sum_{k=1}^{n}\left[\tanh\left(\frac{x+k4i\beta}{2}\right) - \tanh\left(\frac{x-k4i\beta}{2}\right)\right], \qquad (2.15)$$

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where $\xi = m^3/\sin^3 2\beta$. We see that

$$h(x + 2i\beta) - h(x - 2i\beta) \tag{2.16}$$

will be the same as the left side of (2.7) if $\beta = \pi l/(2n+1)$, where l is any integer such that β is in between zero and $\pi/2$. Here it should be mentioned that we have an arbitrariness in our choice of $h(x)$, it is defined up to a constant, but in the end that constant would have to be removed. Then in that case the equations of motion can be expressed like

$$J(x + 2i\beta) - J(x - 2i\beta) = 0\,, \qquad x \in [-a, a]\,, \tag{2.17}$$

where x is in between a and $-a$ and where $J(x) = g(x) - h(x)$, to write it more explicit

$$\begin{aligned} J(x) &= i\xi \cosh x - \frac{S}{4N} \sum_{k=1}^{n} \left[\tanh\left(\frac{x + k4i\beta}{2}\right) - \tanh\left(\frac{x - k4i\beta}{2}\right) \right] + \\ &+ S(w(x - 2i\beta) - w(x + 2i\beta))\,. \end{aligned} \tag{2.18}$$

A small comment before proceeding, the Lagrangian from which the equation of motion was derived only contained a potential term so that the force on a probe eigenvalue $\lambda_i = x$ is equal to the derivative of the potential with respect to x which is nothing other than

$$f(x) = -J(x + 2i\beta) + J(x - 2i\beta)\,. \tag{2.19}$$

Keep this in mind because it will be of use when the superpotential is derived. We can analytical continue $J(x)$ into the complex plane, and it is clear that $J(z)$ is periodic with period $2\pi i$ and in the strip $|Imz| < \pi$, it is holomorphic besides the two cuts at $Rez \in [-a, a]$ and $Imz = \pm 2i\beta$ and simple poles at $z_k = 4ik\beta - i\pi + i2\pi j$, where k is an integer between 1 and n and the integer j is chosen such that z_k is in the interval $-\pi$ and π. Another way to write z_k which might be more illuminating is

$$z_k = \pi \frac{2k - 1}{2n + 1} (-1)^{[\frac{k-1}{m}]} (-1)^{n+1}\,, \tag{2.20}$$

where the bracket [] stands for the integer part of what is inside. In particular if we glue everything together we get a torus with poles in it and it should be possible to express it then as an elliptic function. An elliptic function is determined by its poles and the asymptotic behavior. The asymptotic behavior we can see from $h(z)$ both when we let $z \to \pm\infty$. h is symmetric in z.

$$J = i\frac{\xi}{2} e^{\pm z}\,. \tag{2.21}$$

The structure of our equations are very much the same as in [8] and we could use similar techniques in order to solve the problem, the main differences is that here J is symmetric around $z = 0$ and instead of a double pole at plus infinity, we have one simple pole at plus infinity and one at minus infinity and also several other poles coming pairwise. J is a doubly periodic function, one period coming from going around one cut and the other from going from one cut to the other, thus we can parametrize our system with u such that

$$\begin{aligned} z(u + 2\omega) &= z(u)\,, \qquad z(u + 2\omega') = z(u) + 4i\beta \\ J(u + 2\omega) &= J(u)\,, \qquad J(u + 2\omega') = J(u)\,. \end{aligned} \tag{2.22}$$

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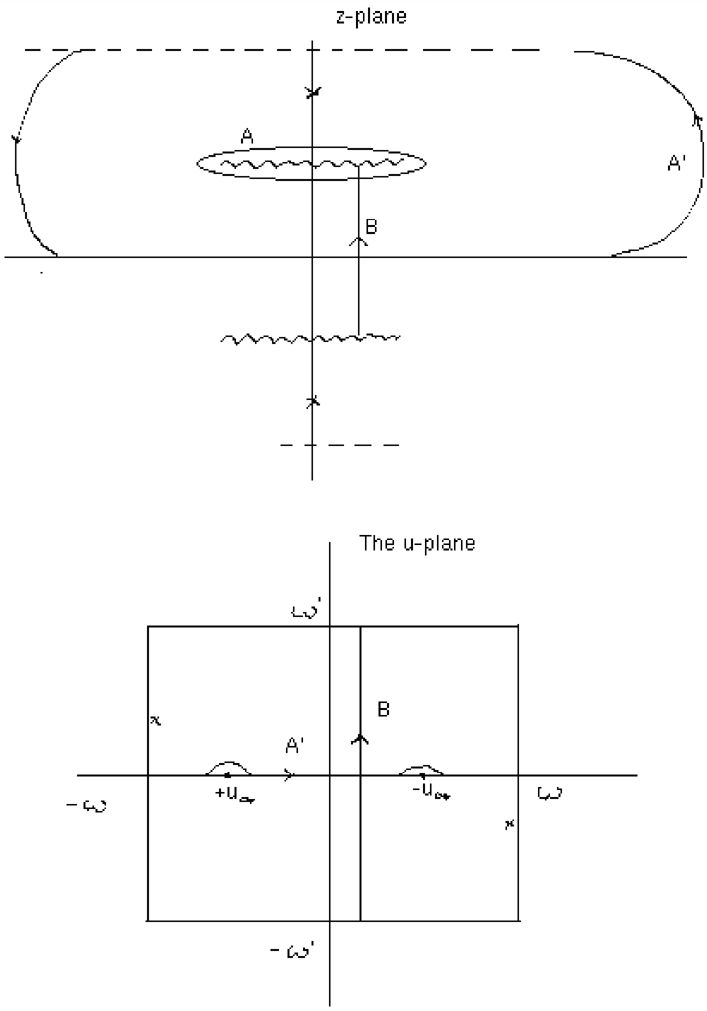

Figure 1: Here you can see the z-plane and the u-plane and how the loops are mapped. The imaginary axis between the cuts in the z-plane are mapped to the imaginary axis in the u-plane, where $\omega = 0$ and the imaginary axis above the cut in the z-plane is mapped to the line $u = \pm\omega$ and $z = \pm 2\beta$ goes to $u = \pm\omega'$.

From [8] we get a regular map between z and u that satisfies the conditions (2.22)

$$\exp z(u) = \frac{H(u_\infty + u)}{H(u_\infty - u)} = \frac{\theta_1(\pi u/2\omega - \beta + \pi/2)}{\theta_1(\pi u/2\omega + \beta + \pi/2)} \tag{2.23}$$

if $u_\infty = (\pi - 2\beta)K/\pi$ where K is the standard quarter period and coincide with ω (see figure 1 to see the z-plane and corresponding u-plane). Sometimes we will use the notation $\beta' = \beta - \pi/2$, because it is practical when dealing with the θ_1 functions. We use the following Ansatz for the Elliptic function J

$$J(u) = A + \frac{B}{\wp(u) - \wp(u_\infty)} + \sum_{u_k} \frac{C_k}{\wp(u) - \wp(u_k)}\,, \tag{2.24}$$

where u_k is the value of u that corresponds to the poles z_k, thus this elliptic function has simple poles at $u = \pm u_\infty$ and $u = \pm u_k$, there are a lot of equivalent ways of doing this ansatz. The constants C_k can be determined through the residues at $\pm z_k$ in equation (2.15) and the constants A and B can be determined by an expansion around u_∞. First expand the expression (2.24) and then do the same for the original J expressed in terms of the theta functions. See appendix for further details, there the following expressions for the constants are obtained

$$\begin{aligned} B &= -i\frac{\xi}{2}\wp'(u_\infty)\frac{H(2u_\infty)}{H'(0)} \\ A &= \frac{B\wp''(u_\infty)}{\wp'^2(u_\infty)} - \frac{C}{\wp(u_\infty) - \wp(u_k)} + i\frac{\xi}{2}\frac{H'(2u_\infty)}{2H'(0)} \\ C &= \frac{S}{2N}\wp'(u_k)\left(z'(u)\right)^{-1} . \end{aligned} \tag{2.25}$$

2.1 The superpotential

Now we would like to apply the methods of Dijkgraaf and Vafa to obtain the effective superpotential. The effective superpotential has the following form:

$$W_{\rm eff}(S) = N\frac{\partial F_0}{\partial S} - 2\pi\, i\tau_0 S\,, \qquad \delta W_{\rm eff}(S) = 0\,, \tag{2.26}$$

where $\tau_0 = \theta/2\pi + i4\pi/g_{YM}^2$ is the gauge coupling, physics is invariant under a change of $\theta \to \theta + 2\pi$. This also has to be extremized in order to get the effective potential, for the Leigh-Strassler deformation, as a function of the coupling constant τ_0. The great thing is that these terms in the expression for the effective superpotential can be expressed in terms of loop integrals over the loops A and B, see figure. From the expression for $J(z)$ (see (2.18)) it is clear that S can be expressed in terms of a line integral around one of the cuts of $J(z)$.

$$2\pi i S = \Pi_A = \int_{C_A} J(z)dz = \int_{-\omega+\omega'}^{\omega+\omega'} J(u)z'(u)du\,. \tag{2.27}$$

Here we have to be a bit careful because there are also the singularities at $\pm z_k$, (see figure and equations (2.20) and (2.15)), with residues $S/2N$ up to a sign, thus integrating around the A' loop instead leads to

$$2\pi i S\left(1 + \frac{n'}{N}\right) = \Pi'_A = \int_{C_{A'}} J(z)dz = \int_{-\omega}^{\omega} J(u)z'(u)du\,, \tag{2.28}$$

where

$$n' = \sum(-1)^{[\frac{k-1}{l}]} = \pm\frac{1}{2}\text{rem}\left(\frac{2l-n}{2l}\right), \tag{2.29}$$

there is a positive sign in front if n is odd and a negative sign if n is even, and $2n$ is the numbers of poles, (this is the same n which appeared earlier e.g. in the expression for β). The contribution n' you get from the poles in the upper half z-plane. Observe that for $l = 1$ and n even, n' is zero.

The other part $\partial_S F_0$ is the derivative of the planar free energy and can be calculated from integrating the force on the probe eigenvalue (2.19) from the infinity to the eigenvalue cut, but also here we should be careful, just as in the other Leigh-Strassler deformation studied by Dorey et.al., there is also a "zero point" contribution to the free energy. From considering the lagrangian (2.6), in the exponent there was the term $2\sinh^2(x/2)$ this could have been rewritten like $(\cosh x - 1)$, that is we get a zero point energy contribution from that constant, which more explicitly looks like $-2Nm^3/g_s \sin^2 2\beta$, the zero genus free energy is this times g_s^2, that is $-2Sm^3/\sin^2 2\beta$. Thus the total derivative of the free energy with respect to S becomes

$$\frac{\partial F_0}{\partial S} = \int_\infty^a f(x) - 2\frac{m^3}{\sin^2 2\beta}\,. \tag{2.30}$$

The first part can be rewritten as an integral over $J(z)$ going from the end of the upper cut to infinity and then back from infinity to the end of the lower cut, and can in turn be transformed to a line integral going directly from the upper cut to the lower. The singularities in the z-plane is all placed along the imaginary axis thus there are no problems deforming the integral.

$$\int_\infty^a f(x) = \Pi_B = \int_{C_2} J(z)dz = \int_{-\omega'}^{\omega'} J(u)z'(u)du\,. \tag{2.31}$$

One thing you could have been worried about concerning this deformation is closing the loop at infinity. In the u coordinates the integrand above, $J(u)z'(u)$, can be expanded around the point in u corresponding to the infinity, there it will consist of a double pole and thus the residue will be zero, thus there is no problem in closing at infinity for the $J(z)$, which we constructed. An extra constant in J would have to be removed in order for this to work.

We see that the effective superpotential expressed in terms of the loop integrals:

$$W_{\text{eff}}(S) = N\Pi_B - \frac{\tau_0 \Pi_{A'}}{1+n'/N} - 2\frac{Nm^3}{\sin^2 2\beta}\,. \tag{2.32}$$

Then extremizing the superperpotential

$$\delta W_{\text{eff}} = N\delta\Pi_B - \frac{\tau_0 \delta\Pi_{A'}}{1+n'/N} = 0\,. \tag{2.33}$$

The loop integrals have some nice properties and the loop integral around B can be expressed as (see appendix)

$$\Pi_B = \left(\tau + \frac{1}{\pi}\frac{\sum v_k}{N-n'}\right)\Pi_A + 2\xi\frac{\theta(2\beta)}{\theta'(0)} \tag{2.34}$$

and

$$\delta\Pi_B = \left(\tau + \frac{1}{\pi}\frac{\sum v_k}{N-n'}\right)\delta\Pi_A \tag{2.35}$$

here we introduced the notation $v_k = \pi u_k/2\omega$. This gives us the relation between the gauge coupling and the elliptic parameter τ

$$\tau\left(N+n'\right) + \frac{1}{\pi}\sum v_k = \tau_0 \tag{2.36}$$

and the effective superpotentials values at the different τs

$$W_{\rm eff} = 2N\xi \frac{\theta(2\beta|\tau)}{\theta'(0|\tau)} - 2N\xi \sin 2\beta\,. \tag{2.37}$$

Here τ should satisfy the solution above (2.36), but because of the non-trivial τ dependence which v_k has we will look at two different limits. The summation over the v_k will be a constant when τ is not too small or/and when β goes to zero, which can be seen to be

$$z_k = \ln \frac{\theta_1(\beta' - v_k)}{\theta_1(\beta' + v_k)} = \ln \frac{\sin(\beta' - v_k)}{\sin(\beta' + v_k)} + \sum_1^{\infty} \frac{1}{n} \frac{q^{2n}}{1 - q^{2n}} \sin 2nv_k \sin 2n\beta' \tag{2.38}$$

q will be very small for most τ's and then the last sum will be negligible. Two cases can occur, either the v_k are pure imaginary or $v_k = i\alpha \pm \pi/2$. Then the relation between v_k and z_k can be expressed like

$$\tan v_k = \cot\beta \tanh \frac{z_k}{2}\,. \tag{2.39}$$

In this case τ becomes

$$\tau = \frac{1}{N - n'}(\tau_0 - j - i\alpha) \tag{2.40}$$

where j is an integer or a half integer, if it is an integer it will correspond to a shift by θ with 2π and α just the sum over all the imaginary values in u_k thus the effect is that τ_0 gets shifted by a constant and also N get shifted. In the limit β goes to zero in such a way that n will be an even number, τ will become $\tau = \tau_0/N$ and equation (2.37) will coincide with the mass perturbed case calculated by Dijkgraaf, Vafa [3] and Dorey [5].

How v_k behaves for small values of τ can be seen from using the modular properties of the θ_1 functions

$$\begin{aligned} z_k &= \ln \frac{\theta_1(\beta' - v_k|\tau)}{\theta_1(\beta' + v_k|\tau)} = \ln \frac{e^{-i(v_k-\beta')^2/\pi\tau}\,\theta_1((\beta' - v_k)/\tau| - 1/\tau)}{e^{-i(v_k+\beta')^2/\pi\tau}\,\theta_1((\beta' + v_k)/\tau| - 1/\tau)} \\ &\approx 4iv_k\beta'/\pi\tau + \ln \frac{\sin((\beta' - v_k)/\tau)}{\sin((\beta + v_k)/\tau)}\,. \end{aligned} \tag{2.41}$$

We have to look at two different cases, the case when v_k is purely imaginary, then the second term becomes $2iv_k/\tau$ and in the other case the second term becomes $2i\beta/\tau$. In the first case then

$$4iv_k \approx \tau\pi\frac{z_k}{\beta} = \tau\pi\frac{(2k-1)}{l}(-1)^{[\frac{k-1}{m}]}(-1)^{n+1} \equiv \tau\pi L_k\,. \tag{2.42}$$

And the second case we have

$$\begin{aligned} 4iv_k &= \frac{\pi\tau}{\beta - \pi/2}(z_k - i\pi(-1)^p) - 2\pi i \\ &= 2\pi\tau\left[\frac{2k - 1 - (2n+1)}{2l - 2n - 1}\right](-1)^{[\frac{k-1}{m}]}(-1)^{n+1} - 2\pi i(-1)^p \\ &\equiv \pi\frac{\tau}{2}M_k - 2\pi i(-1)^p\,, \end{aligned} \tag{2.43}$$

where p is an even number if z_k is positive and odd if z_k is negative. For the expression of z_k see equation (2.20). The following relation holds between τ and τ_0

$$\tau = \frac{(\tau_0 + j)}{(N - n' + \sum_k L_k/4 + \sum_k M_k/2))}. \tag{2.44}$$

The second summation is over the k which fulfill the condition

$$|2k - 1| > 2l \tag{2.45}$$

and the first summations over the other k's. So we see that the effect is that τ_0 get shifted with an half integer or an integer and N gets shifted with something that sometimes is an integer and for some special values of β it could be zero.

To get the gluino condensate we differentiate the effective superpotential (2.37) with respect to τ_0, that will also correspond to the S we calculated in the appendix. Remember though to see that you have to remember that the relation between τ_0 and τ is in general a bit complicated. Anyway taking the mass decoupling limit, letting $m \to \infty$ together with $\tau \to i\infty$ such that $m^3 q^2 = \Lambda^3$, where Λ is a constant, we see that we get that the effective superpotential becomes a constant independent of β as it should.

3. Classical vacua, confinement and S-duality

In [9] they found that the underlying S-duality from the $N = 4$ SYM was realized, in the mass perturbed case, via modular transformation on the gauge coupling relating the superpotential for different massive vacua and a similar thing was found in [7]. In [7] they extract an SU(N) version of the Leigh-Strassler deformation they looked at and found for instance that the confining vacuum was related to the higgs vacuum via S-duality. Here we will look at the eigenvalue distribution around the classical $\Phi_{cl} = 0$ and see what will happened to it if we perform an S-transformation and then take its classical limit. We will see that it is proportional to some of the classical vacua of the theory.

Berenstein et.al. [10] looked at the classical solutions to the same model as we are considering. That is they started from the superpotential

$$W = \mathrm{Tr}\left(\phi_1\phi_2\phi_3 - q\phi_1\phi_3\phi_2 + m\sum \phi_i^2\right). \tag{3.1}$$

The F-flatness condition gave them the following relation between the fields

$$[\phi_j, \phi_{j+1}]_q = \phi_{j+2}\,, \qquad \text{cyclic on } j, \text{ mod } 3 \tag{3.2}$$

from which they derived some of the vacua of the theory. The q-commutator is defined through $[a, b]_q = ab - qba$. They did a field redefinition in order to get rid of the m, in the same fashion a field redefinition can be made in order to get the commutator in the form we have. Let $\phi \to q^{1/2}\phi$, then the commutator becomes

$$q^{-1/2}\phi_i\phi_j - q^{1/2}\phi_j\phi_i\,. \tag{3.3}$$

Anyway in order to get the vacua they had to find irreducible representations to this algebra. A certain class of these representations will be deformed $sl(2,C)$ representations, thus in the case $q=1$ the algebra takes the form of $sl(2,C)$. First they noted that there exist a one dimensional representation for $q \neq 1$ looking like

$$\phi_j = \frac{1}{1-q} \tag{3.4}$$

and a two dimensional irreducible representation looking like

$$\phi_j = \frac{-i}{1+q}\sigma_j\,, \tag{3.5}$$

where the σ_j are the Pauli matrices. This latter one can be looked upon as a deformation of $sl(2,C)$ representation. Then they came up with an ansatz to find a general form of a deformed $sl(2,C)$ representation. In the case of even dimension $2p$ of the representation they found the following eigenvalues to the matrices ϕ_i

$$\pm\alpha_n = \pm\frac{1}{q^{p-n}(q^{-1/2}+q^{1/2})}\sigma_{2(p-n)}[q]\,, \tag{3.6}$$

where $\sigma_x[q] = 1+q+q^2+\ldots+q^x$, here we have accounted for the rescaling of the field ϕ. In the case $q=e^{2i\beta}$ this can in fact be rewritten as

$$\pm\,\alpha_n = \pm\frac{\sin(\beta(2(p-n)-1))}{\sin 2\beta}\,. \tag{3.7}$$

And in the case of odd dimension $2p+1$ of the representation, the eigenvalues they obtained look like

$$0,\ \pm\alpha_n = \pm\frac{\sigma_{(p-n)}[q^2]}{q^{p-n}}\,. \tag{3.8}$$

They warn that these solutions are not D-flat, but the solutions is related to a D-flat solution via an $SL(M)$ transformation.

In [7] they conjecture a method for getting the values for the eigenvalues of the condensate of the field Φ, we will see that this method indeed gives us the right type of expectation value of Φ^2, and performing an S-transformation on those eigenvalues reproduces something proportional to the vacua above in the classical limit $\tau \to i\infty$. In order to get the eigenvalue distribution the idea is now first to notice that the eigenvalues λ_i to the field Φ_1 is

$$\lambda_i = 2m\frac{\sinh(\mu_i/2)}{\sin 2\beta}\,, \tag{3.9}$$

where μ_i is between $-a$ and a. The cut in the resolvent that emerges in J is displaced from the position of the eigenvalues μ_i with a distance $i2\beta$ if we are considering the upper cut in the z plane for J. Their proposal is then to displace μ_i with a distance $i2\beta$ and then evaluate the function $\lambda(x)$ where the i's have been made continuous along the upper cut (the A-cycle)

$$\begin{aligned}\lambda(x) &= 2m\frac{\sinh\frac{(x-2i\beta)}{2}}{\sin 2\beta} = \frac{m}{\sin 2\beta}\left(e^{i\beta}\frac{\theta_1^{1/2}(\pi x+\tau\pi/2-\beta)}{\theta_1^{1/2}(\pi x+\tau\pi/2+\beta)} - e^{-i\beta}\frac{\theta_1^{1/2}(\pi x+\tau\pi/2+\beta)}{\theta_1^{1/2}(\pi x+\tau\pi/2-\beta)}\right)\\ &= \frac{m}{\sin 2\beta}\left(\frac{\theta_4^{1/2}(\pi x-\beta|\tau_0/N')}{\theta_4^{1/2}(\pi x+\beta|\tau_0/N')} - \frac{\theta_4^{1/2}(\pi x+\beta|\tau_0/N')}{\theta_4^{1/2}(\pi x-\beta|\tau_0/N')}\right) \qquad x\in[0,1]\end{aligned} \tag{3.10}$$

here N' stands for the shifted N we got when solving the equations of motions. We will not be so careful about the shift and just considering N' as if it were the usual N, a more careful treatment should bee done, but now we look at things very roughly. It is easy to check in the classical limit $\tau_0 \to i\infty$ the expression above goes to zero, which is expected because we looked at quantum fluctuation around the classical eigenvalue zero. Then they claim that the eigenvalues should be uniformly distributed along this and the expectation value of the field squared should be given by integrating the square of this over the range of x gives.

$$\langle \lambda^2(x) \rangle = N \int_0^1 \lambda(x) = \frac{2Nm^2}{\sin^3 2\beta} \frac{\theta(2\beta)}{\theta'(0)} - \frac{2Nm^2}{\sin^2 2\beta}\,. \tag{3.11}$$

Again the last part is just due to the zero energy. Here we see that this at least are at agreement with the expression of the effective superpotential (2.37), to differentiate the superpotential with respect to the mass should give as the expectation value of the field Φ^2.

Now we will see which nice properties this $\lambda(x)$ has under S-transformation, letting $\tau_0 \to -1/\tau_0$ the eigenvalues $\lambda(x)$ becomes

$$\lambda(x) = \frac{m}{\sin 2\beta}\left(\frac{\theta_3^{1/2}(\pi x - \beta| - 1/\tau_0 N')}{\theta_3^{1/2}(\pi x + \beta| - 1/\tau_0 N')} - \frac{\theta_3^{1/2}(\pi x + \beta| - 1/\tau_0 N')}{\theta_3^{1/2}(\pi x - \beta| - 1/\tau_0 N')}\right) \qquad x \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{3.12}$$

Now we like to see what will happened with this in the classical limit $\tau \to i\infty$. It is easy to take the limit if we use the modular transformation rules of the theta function

$$\theta_3(x|\tau) = (-i\tau)^{1/2} e^{-ix^2/\pi\tau}\theta_3\left(\frac{x}{\tau}| - \frac{1}{\tau}\right) \tag{3.13}$$

using this we get that the $\lambda(x)$ equals

$$\lambda(x) = \frac{m}{\sin 2\beta}\left(e^{-2i\beta x\tau N}\frac{\theta_3^{1/2}(\tau_0 N(\pi x - \beta)|\tau_0 N)}{\theta_3^{1/2}(\tau_0 N(\pi x + \beta)|\tau_0 N)} - e^{2i\beta x\tau N}\frac{\theta_3^{1/2}(\tau_0 N(\pi x + \beta)|\tau_0 N)}{\theta_3^{1/2}(\tau_0 N(\pi x - \beta)|\tau_0 N)}\right). \tag{3.14}$$

The theta three's goes to one in the limit $\tau \to i\infty$. Thus after doing an S-transformation on $\lambda(x)$ wee see that it becomes in the classical limit

$$\lambda(x) = im\frac{\sin(2\beta x\tau N)}{\sin 2\beta}\,. \tag{3.15}$$

We could say that also β are supposed to transform under S like $\beta \to \beta/\tau_0$ [7] then the above expression can be written like

$$\lambda(x) = im\frac{\sin 2\beta}{\sin 2\beta/\tau}\frac{\sin(2\beta x N)}{\sin 2\beta}\,. \tag{3.16}$$

This expression is up to the strange factor in front of it equal to the classical eigenvalues (3.7) if you make a discretization of x.

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4. Conclusions

First of all an exact solution to the Leigh-Strassler deformation under consideration was found and was shown to have of an elliptic structure. The solution was found for special values of the parameter β, which parametrize the deformation, these β's were the ones that could be written as some fraction of π. The solution was used to apply the method proposed by Dijkgraaf and Vafa, to find an effective superpotential for the Leigh-Strassler deformation and thus the gluino condensate. It looks very similar to the effective potential of the other Leigh-Strassler deformation which has been studied in this contexts, but there are some crucial differences as the symmetry around the expectation value of the field Φ and its elliptic parameter τ has a more complicated dependence on the gauge coupling constant τ_0, which might be of interest. For some special β you get the same simple τ_0 dependence as in the earlier studied cases.

Then we had a brief look at how S-transformation acts on the solution and saw that the confining vacuum was related to some higgs vaccum in the classical limit.

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A. Calculations

For all the relations between elliptic functions and theta functions and the integrals over them see [11] and [12]

A.1 The coefficients

The constants C_k can be determined by the fact that the residue of (2.15) around u_k should coincide with the residual of (2.24) around u_k. Do a Laurent expansion of tanh around $x = z_k$ in (2.15)

$$\frac{S}{4N} \tanh\left(\frac{x - z_k}{2}\right) = \frac{S}{2N} \frac{1}{x - z_k} . \tag{A.1}$$

Then using $x = z(u)$, where $z(u)$ is given by 2.23 and expanding it around u_k and put it in the equation above

$$\frac{S}{2N} \frac{1}{z'(u)(u - u_k)} . \tag{A.2}$$

Expanding (2.24) around u_k gives

$$\frac{C}{\wp'(u_k)} \frac{1}{u - u_k} . \tag{A.3}$$

And from putting the residuals equal C becomes

$$C = \frac{S}{2N} \wp'(u_k) \left(z'(u)\right)^{-1} . \tag{A.4}$$

Now we can decide the constants A and B through doing an expansion around u_∞. First do it for the expression (2.24)

$$J(u) = A + \frac{B}{\wp'(u_\infty)(u-u_\infty)} - \frac{B\wp''(u_\infty)}{2\wp'^2(u_\infty)} + \frac{C}{\wp(u_\infty)-\wp(u_k)} \tag{A.5}$$

If we use equation (2.21) for J and put in the expression for $z(u)$ in terms of theta functions we get the behaviour at infinity as

$$J(u) = i\frac{\xi}{2}\left(\frac{H(2u_\infty)}{H'(0)(u_\infty-u)} - \frac{H'(2u_\infty)}{H'(0)}\right). \tag{A.6}$$

From here we get A and B expressed in terms of theta and elliptic functions

$$\begin{aligned} B &= -i\frac{\xi\wp'(u_\infty)}{2}\frac{H(2u_\infty)}{H'(0)} \\ A &= \frac{B\wp''(u_\infty)}{\wp'^2(u_\infty)} - \frac{C}{\wp(u_\infty)-\wp(u_k)} - i\xi\frac{H'(2u_\infty)}{H'(0)}. \end{aligned} \tag{A.7}$$

A.2 The integral calculations

The two integrals of interest in order to get the effective superpotential is:

$$\Pi_{A'} = \int_{C_{A'}} J(z)dz = \int_{-\omega}^{\omega} J(u)z'(u)du\,. \tag{A.8}$$

And

$$\begin{aligned} \Pi_B &= \int_{C_B} J(z)dz = \int_{-\omega'}^{\omega'} J(u)z'(u)du \\ z'(u) &= \frac{H'(u+u_\infty)}{H(u+u_\infty)} - \frac{H'(u-u_\infty)}{H(u-u_\infty)} = \zeta(u+u_\infty) - \zeta(u-u_\infty) - 2\zeta(\omega_1)\frac{u_\infty}{\omega} \\ &= -\frac{\wp'(u_\infty)}{\wp(u)-\wp(u_\infty)} + 2\zeta(u_\infty) - 2\zeta(\omega_1)\frac{u_\infty}{\omega} = -\frac{\wp'(u_\infty)}{\wp(u)-\wp(u_\infty)} + \frac{H'(2u_\infty)}{H(2u_\infty)} - \frac{\wp''}{2\wp'}(u_\infty)\,. \end{aligned}$$

The integral can be divided in the following pieces, which can be rewritten in standard elliptic integral forms. First the piece consisting of the poles u_∞ having the constant B in it

$$\begin{aligned} \Pi_1 &= -B\wp'(u_\infty)\int\frac{1}{(\wp(u)-\wp(u_\infty))^2} + B\left(\frac{H'(2u_\infty)}{H(2u_\infty)} - \frac{\wp''}{2\wp'}\right)\int\frac{1}{\wp(u)-\wp(u_\infty)} \\ &= -B\left[\frac{\wp''(u_\infty)}{\wp'^2}\ln\frac{\sigma(u+u_\infty)}{\sigma(u-u_\infty)} - \frac{1}{\wp'(u_\infty)}(\zeta(u+u_\infty)+\zeta(u-u_\infty)) - \right. \\ &\quad - \left(\frac{2\wp(u_\infty)}{\wp'(u_\infty)} + \frac{2\wp''(u_\infty)\zeta(u_\infty)}{\wp'^2(u_\infty)}\right)u + \frac{1}{\wp'}\left(\frac{\wp''}{2\wp'} - \frac{H'(2u_\infty)}{H(2u_\infty)}\right)\times \\ &\quad \left.\times\left(\ln\frac{\sigma(u-u_\infty)}{\sigma(u+u_\infty)} + 2u\zeta(u_\infty)\right)\right] \end{aligned} \tag{A.9}$$

and the part consisting of C

$$\begin{aligned}\Pi_2 &= C\left(\frac{-\wp'(u_\infty)}{\wp(u_k)-\wp(u_\infty)}+\frac{H'(u_\infty)}{H(u_\infty)}\right)\int\frac{1}{\wp(u)-\wp(u_k)}+\\ &\quad+\frac{1}{\wp(u_k)-\wp(u_\infty)}\int\frac{C\wp'(u_\infty)}{\wp(u)-\wp(u_\infty)}\\ &= -\frac{S}{2N}\left[\ln\frac{\sigma(u-u_k)}{\sigma(u+u_k)}+2u\zeta(u_k)\right]+\frac{C}{\wp(u_k)-\wp(u_\infty)}\left[\ln\frac{\sigma(u-u_\infty)}{\sigma(u+u_\infty)}-2u\zeta(u_\infty)\right].\end{aligned}$$

The only contribution of the integral over the constant A comes from the C_2 curve

$$\Pi_3 = A\int_{C_2}dz = 4i\beta\left(\frac{B\wp''(u_\infty)}{2\wp'^2(u_\infty)}-\frac{C}{\wp(u_\infty)-\wp(u_k)}-i\xi\frac{H'(2u_\infty)}{2H'(0)}\right). \tag{A.10}$$

The terms in Π_A and Π_B linear in u will just relate the integrals with a factor of $\tau=\omega'/\omega$, that is $\Pi_B=\tau\Pi_A$ (linear in u). From looking at the integrands we see that we need to know

$$\begin{aligned}\left[\ln\frac{\sigma(u+u_\infty)}{\sigma(u-u_\infty)}\right]_{-\omega}^{\omega} &= \ln\frac{\sigma(\omega+u_\infty)}{\sigma(\omega-u_\infty)}-\ln\frac{\sigma(-\omega+u_\infty)}{\sigma(-\omega-u_\infty)}\\ 2\ln\frac{\sigma(\omega+u_\infty)}{\sigma(\omega-u_\infty)} &= 2\ln\frac{\theta_1(\pi/2-\beta+\pi/2)}{\theta_1(\pi/2+\beta-\pi/2)}+4\eta\frac{\beta}{\pi}=4\eta\frac{\beta}{\pi}\end{aligned} \tag{A.11}$$

and

$$\left[\ln\frac{\sigma(u+u_\infty)}{\sigma(u-u_\infty)}\right]_{-\omega'}^{\omega'}=2\ln\frac{\theta_1(\tau\pi/2-\beta+\pi/2)}{\theta_1(\tau\pi/2+\beta-\pi/2)}+4\tau\eta\frac{\beta}{\pi}=4i\beta+4\tau\eta\frac{\beta}{\pi}. \tag{A.12}$$

And last we need

$$\begin{aligned}\left[\zeta(u+u_\infty)+\zeta(u-u_\infty)\right]_{-\omega}^{\omega} &= 4\eta\\ \left[\zeta(u+u_\infty)+\zeta(u-u_\infty)\right]_{-\omega'}^{\omega'} &= 4\eta'\,,\qquad \eta'=\tau\eta-\frac{\pi}{2\omega_1}.\end{aligned} \tag{A.13}$$

Now we see that

$$\Pi_B=\tau\Pi_A'+2\xi\frac{\theta(2\beta)}{\theta'(0)}+\frac{S}{2N}\sum 4iv_k\,. \tag{A.14}$$

Using $S=N\Pi_A/((N-n')2\pi i)$ the above becomes

$$\Pi_B=\tau\left(1+\frac{1}{\pi}\frac{\sum v_k}{N-n'}\right)\Pi_A'+2\xi\frac{\theta(2\beta)}{\theta'(0)}\,. \tag{A.15}$$

A.3 Useful relations between elliptic and theta functions

$$\begin{aligned}\zeta(u)-\frac{\zeta(\omega_1)u}{\omega_1} &= \frac{\pi}{2\omega_1}\frac{\theta_1'(\pi u/2\omega_1)}{\theta_1(\pi u/2\omega_1)}\\ \ln\frac{\theta(\alpha-\beta)}{\theta(\alpha+\beta)} &= \ln\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}+\sum_1^\infty\frac{1}{n}\frac{q^{2n}}{1-q^{2n}}\sin 2n\alpha\sin 2n\beta\\ 2\eta(\alpha) &= \eta(2\alpha)-\frac{\wp''(\alpha)}{\wp'(\alpha)}\\ \sigma(z) &= \frac{2\omega}{\pi}\exp\frac{\eta z^2}{2\omega}\frac{\theta_i(v)}{\theta'(0)}\,.\end{aligned} \tag{A.16}$$

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